

Aggregated threshold functions or how to measure the performance of an electoral system[☆]

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Abstract

This article introduces a set of functions that measures the mechanical performance of an electoral system. Aggregated threshold functions offer the necessary and sufficient share of the vote nationwide to win a given number of seats. Traditionally, electoral systems have been measured taking into account the share of the vote required to win one seat given a district. In the approach used here, the values obtained are calculated taking into account all districts in which a country is divided and for any number of seats. This article offers the definition and formalization of these functions. Once the aggregated threshold functions are defined in all their terms, I show some data validation to test their capacity of prediction. The main goal of the article is to provide with a tool that can be used, for example, to develop a measure that summarizes in a single value the functioning of an electoral system. This value can be used by electoral reformers as well as by students of electoral systems to test the consequences of electoral systems as a whole.

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Keywords: Threshold of representation; Threshold of exclusion; Perfect proportionality; District and national levels

1. Introduction: the research question

How can the mechanical performance of an electoral system be measured as a whole? A key issue when studying the functioning of electoral systems has been to find the threshold of votes that each electoral system requires to win a given number of seats. However, most of the theoretical and empirical research on this subject has been devoted to work out the share of the vote that each electoral

system requires to win just one seat in the parliament. This effort has also been mainly concentrated around working out these thresholds at district level. In this sense, particularly influential at theoretical level have been the early works by Rokkan (1968), Rae et al. (1971) or Lijphart and Gibberd (1977). Empirically, the work by Taagepera and Shugart (1989) and Lijphart (1994) has probably set up the dominant position on the issue.

One can, however, asks the following question, what are the necessary and sufficient shares of votes nationwide to win any number of seats in any electoral system? The studies just mentioned do not answer this question even though it is probably at the center of their interest. The calculation of the necessary and sufficient shares of the vote nationwide to win any number of seats goes beyond these earlier approaches in several ways. Firstly, because it highlights the importance of finding out

[☆] I would like to specially thank Adam Przeworski for all his time, patience and encouragement during my research. Earlier versions of this article have also benefited from the comments and suggestions made by Ignacio Sánchez-Cuenca, Ignacio Lago, Ferran Martínez and Braulio Gómez. I also have to thanks the two anonymous referees whose suggestions have, no doubts, enriched the final version of this article. None of them are responsible for any mistakes this paper may contain.

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a threshold to win not just one seat but any number of them. Secondly, this calculation allows finding out these values for any electoral system no matter its institutional design. Thirdly, a new set of functions that goes beyond district level and pursues nationwide values is sought.

What makes valuable these new functions in relation with all existing measures is that they parametrically summarise the mechanical functioning of any electoral system¹ in a single exclusive value providing, then, with a straight method to measure the performance of electoral systems. As an illustration, an interesting value around which electoral systems might be measured could be, for example, the necessary share of the total vote to win half of the seats in the parliament. This value would allow locating electoral systems in a continuum so that they could be contrasted with, say, an ideal point of perfect proportionality².

This possible way to measure the mechanical functioning of electoral institutions is appealing for both students of electoral systems and also for electoral reformers since they may overcome an interesting paradox. While electoral systems are a recurrent topic in the literature, there is not a unanimous consensus as how to characterize them. In some cases, electoral systems are summarized by categorical variables (Cohen, 1997; Persson and Tabellini, 2003). In these studies, a distinctive feature of an electoral system like the electoral formula is the exclusive criterion used to group electoral institutions around different categories of a variable. Other set of studies (Katz, 1997) use empirical measures, like the effective threshold, that produce not categorical but continuous values. None of these approaches are satisfactory. The use of categorical variables does not allow controlling for specific particularities in the functioning of electoral systems classified under the same category – does the number of districts matter? Is the size of the assembly important to understand some consequences of electoral systems? In the same sense, the use of empirical measures like the effective threshold can only be justified by the goodness of fit with the values empirically observed but they are neither calculated following a clear-cut logic nor do they summarise the mechanical performance of an electoral system as a whole.

Since aggregated threshold functions take into account the interaction of every institutional component of an

electoral system, they provide more information than those variables based on a single category. Furthermore, aggregated threshold functions are calculated following a well-defined logic and offer nationwide values in clear contrasts with other measures like the effective threshold. By using this new measure, students of electoral systems will benefit from using a continuous variable that accounts for each institutional variable that defines an electoral system. Similarly, electoral reformers would find in these functions a convenient tool to test the consequences of different institutional designs. Institutional design experts could observe how a change of the number of districts or a change in the electoral formula, for instance, would affect the functioning of the reformed institution.

The purpose of this article is to define what I call aggregated threshold functions. It is mainly a theoretical formulation of how these functions are modelled and, therefore, questions derived from the specific functioning of aggregated threshold functions are left for future research. However, I will proceed to some necessary empirical work in order to test the validity of the functions. In the following pages, I proceed as follow. First, I give an account of the most inspiring works that first approached the issue at stake and posited some questions that these studies did not answer. Second, I introduce a novel and appealing approach due to Penadés (2000) that offers precisely threshold functions for any number of seats and for any electoral systems for a given district. Thirdly, I reformulate this latter approach in order to make it valid at national level. Finally, data validation are offered to show the working of these aggregated threshold functions.

2. Earlier approaches

The literature has recurrently provided different methods to measure the mechanical performance of electoral systems. Theoretically, the concepts of both the threshold of representation and the threshold of exclusion have centred most of the attention. Rokkan (1968) posited a key question in the study of electoral systems. In his own words, “How little support can possibly earn a party its first parliamentary seat?” (Rokkan, 1968:6–21). This question was sparked by his keen interest in identifying the characteristic range of votes required by each electoral system for a party to win a seat. In other words, the issue at stake was about calculating the best condition to win parliamentary representation under any electoral system. This idea turned into what Rokkan called the threshold of representation, which referred to the minimum share of votes that allowed any party, p , to win one seat in a district. In his work, Rokkan calculated this value for three types of electoral formulae: d’Hondt, Sainte-Laguë and Hare.

¹ In the rest of the article by mechanical performance of an electoral system is meant the particular transformation of votes into seats that each electoral system produces given its own institutional components.

² Let us define an electoral system as perfectly proportional if the share of votes obtained by any given party gives it the same share of the seats in, say, the legislative assembly.

The concept of threshold of representation was contested by Rae et al. (1971). Rather than focusing on the most favourable condition that allowed a party to win a seat, Rae, Hanby and Loosemore were more concerned with the maximum support a party could win and still not obtain a seat. This was the threshold of exclusion, which introduced the idea that a party might not win a seat even though it had strong political support. The threshold of exclusion was calculated for the same three electoral formulae used by Rokkan, as well as for the plurality formula, all applied at district level.

These theoretical developments still leave one question unanswered. One could ask if such thresholds might be calculated not just for one seat but for any number of them in a multi-member district. Rae et al. (1971:485) proposed some possible answers to this question in their work. However, the most rigorous attempt to find such thresholds was due to Lijphart and Gibberd (1977). Here, Lijphart and Gibberd refined the threshold of inclusion and exclusion calculated by Rokkan and Rae et al. and expanded the analysis to a new divisor-based electoral formula: Modified Sainte-Laguë. But their most valuable contribution was to develop a formal reasoning to calculate these values for any number of seats in a given district. This is what they termed “payoffs functions” (Lijphart and Gibberd, 1977: 230).

Empirically, electoral systems have been measured predominantly through a sort of average value between the threshold of representation and the threshold of exclusion. The effective threshold has been mainly developed by Taagepera and Shugart (1989) and later by Lijphart (1994) and it provides with the share of the votes around which a seat can be won in a given district³.

These studies at theoretical and empirical levels are of great value. However, they must be considered with some caution for the following reasons. First, as defined in the literature, the effective threshold is a purely empirical measure which definition does not respond to any logic⁴. It would be desirable, then, to have a unifying and better defined function to measure the performance of any

electoral system. Second, one may still ask if it is possible to define some functions capable of calculating both the threshold of representation and the threshold of exclusion for any electoral system using any electoral formula. As I pointed out, Lijphart and Gibberd’s work only calculated the values for the most commonly used electoral formula: d’Hondt, Sainte-Laguë, Modified Sainte-Laguë and Hare. However, the universe of electoral formulae is broader (Gallagher, 1992), raising the question of whether it is possible to identify a function capable of calculating the best and the worst conditions to win any number of seats in a district when any electoral formula is applied. In other words, can we obtain inclusion and exclusion values using a general and logically defined function that could be applied to any electoral system using any conceivable electoral formula? One excellent and extremely attractive approach to this problem lies in what Penadés (2000) calls threshold functions.

3. Threshold functions

First, it is perhaps necessary to refer to the terminology that will be used from this point on. As noted above, Lijphart and Gibberd used the term “payoff functions” to refer to the share of the vote needed to win any number of seats in a district. However, as Penadés rightly argues, “payoff functions” should be used to refer to those functions that predict the number of seats that a party wins given its share of the vote, whereas threshold functions should refer to a set of functions used to calculate the minimum and maximum proportion of votes needed to win a determined number of seats in a district (Penadés, 2000:35). The term threshold function is used here in this second sense.

Penadés (2000) theoretically refines and enriches the work of previous scholars on the field by calculating a universal formulation for threshold functions. His study proposes threshold functions for both majoritarian and proportional representation electoral systems. These threshold functions apply, then, for any number of seats and for any possible electoral formula in a given district. As in Lijphart and Gibberd, Penadés’ threshold functions make it possible to calculate the necessary and sufficient shares of the vote that a party must obtain in order to win a determined number of seats given any elemental electoral system. An elemental electoral system comprises three components: an electoral formula, F , a district magnitude, M_d , and finally the number of competing parties in that district, P_d (Penadés, 2000:23).

A general description about how threshold functions are conceived can illuminate this discussion. One way of approaching these functions is by working out the

³ A full discussion on the effective threshold can be found in Lijphart (1994): 25–30. See also Katz (1997).

⁴ Taagepera and Shugart (1989) used, for example, the divisors-based d’Hondt electoral formula to calculate the exclusion threshold and the quota-based Hare electoral formula to calculate the threshold of representation. And while agreeing with this method of calculating the exclusion threshold, Lijphart (1994) disagreed with them with respect to the appropriate method of calculating the inclusion threshold. Instead of using a largest remainder method with the Hare quota, he proposed half of it. While it is true that all these measurements served Taagepera and Shugart’s and Lijphart’s purposes well, they provide no mechanisms or logical procedures to explain why this is so.

necessary and the sufficient share of the vote to obtain the same number of seats (Rae et al., 1971; Lijphart and Gibberd, 1977; Penadés, 2000). It should be remembered that the threshold of representation refers to the share of the vote below which it is impossible to obtain representation. In this sense, the minimum proportion of votes required to win one seat is also the necessary share of the vote to obtain representation. If a party, p , wins a share of the vote which equals the necessary share of votes to obtain a given number of seats in district d , S_d^p , that party will have a chance of winning that amount of seats in that district. Likewise, the threshold of exclusion is the share of votes at which it is impossible for a party not to obtain representation. That is to say, the sufficient share of the vote to win one seat equals the maximum share of votes with which a party will obtain no seats. If a party, p , obtains a share of the votes which is above the sufficient share of votes required to win S_d^p seats, that party will get at least that amount of seats in district, d .

This reasoning gives us two important definitions that should make it possible to fully understand threshold functions.

Definition 1. $V_{S_d}^{nec}(F, M_d, P_d)$: This function determines the necessary, but not sufficient, share of the vote in order to obtain any number of seats in district d , S_d , provided that S_d is a number smaller or equal to district magnitude, $1 \leq S_d \leq M_d$ and given the electoral formula, F , the magnitude of district d , M_d , and the number of competing parties in that district, P_d .

Definition 2. $V_{S_d}^{suf}(F, M_d, P_d)$: This function determines the sufficient number of votes to obtain S_d seats given F , M_d and P_d and provided that $1 \leq S_d \leq M_d$ (Penadés, 2000:127).

From these definitions, we derive the following conclusions for any party, p , with a proportion of votes, V_d^p , given F , M_d and P_d :

Conclusion 1. If $V_d^p \leq V_{S_d}^{nec}(F, M_d, P_d)$, then $S_d^p \leq S_d$.

Conclusion 2. If $V_d^p \geq V_{S_d}^{suf}(F, M_d, P_d)$, then $S_d^p \geq S_d$.

Conclusion 3. If $V_{S_d}^{nec}(F, M_d, P_d) \leq V_d^p \leq V_{S_d}^{suf}(F, M_d, P_d)$, then $\max(S_d^p) = S_d$ provided⁵ that $V_{S_d+1}^{nec} \geq V_{S_d}^{suf}$.

In words, for a given district, if party p wins a share of the vote below the necessary share of the vote required to win one seat, that party will not win that seat. If, on the contrary, party p wins a share of votes

higher than the sufficient share of the vote to win one seat, then party p will win at least a seat in that district. However, if party p wins a share of the vote which is contained in the interval formed by the necessary and the sufficient shares of the vote to win one seat, then, that party will win at most one seat in that district.

Threshold functions comprise, therefore, two different functions: on the one hand, the function of necessary votes for any number of seats, $V_{S_d}^{nec}(F, M_d, P_d)$, and on the other hand, the function of sufficient votes for those S_d seats, $V_{S_d}^{suf}(F, M_d, P_d)$. The mathematical development of these functions goes as follows.

As already said, threshold functions are calculated by Penadés at district level and on the basis of two specific elements of the electoral formula: the modifier of the quota, n , for quota-based electoral formulae⁶ and the adjustment term, c , for divisor-based electoral formulae⁷. Given the existence of these two families of electoral formulae, two approaches will also be used to calculate threshold functions⁸. The first is based on n , the modifier of the quota, and the second on the adjustment term, c .

⁶ Briefly, a quota, $Q(n)$, is defined as $Q(n) = (V_d)/(M_d + n)$ where M_d is the magnitude of district d , n is the modifier of the quota and V_d corresponds to the total of valid votes in district d . When $n = 0$, the Hare quota or simple quota is obtained; when $n = 1$, the Droop quota and when $n = 2$, the Imperiali quota is obtained. The size of n is important. The largest its value, i.e., the smaller the quota, the less proportional the electoral formula becomes. For a detailed account of this approach see Taagepera and Shugart (1989):30 and Penadés (2000):55–82.

⁷ Summarizing, divisors-based electoral formulae are defined around the concept of a constant non-negative divisor criterion. By this, it is meant an adjustment rule,

$$c(S_d^p) = S_d^p + c \text{ for } S_d^p > 0$$

where c is the adjustment term and S_d^p refers to the number of seats that party p wins in district d . This rule is pre-established for each electoral formula. So, for example, for d'Hondt, the criterion is $c(S_d^p) = S_d^p + 1$. This means that in order to get S_d^p seats, party p must fulfil the following restriction

$$S_d^p \geq \frac{V_d^p}{X} \geq S_d^p + 1$$

where X is a non pre-established divisor that must be found in order to allocate the M_d seats. (see Balinsky and Young, 1982). For Sainte-Laguë $c(S_d^p) = S_d^p + 0.5$ or

$$S_d^p - 0.5 \geq \frac{V_d^p}{X} \geq S_d^p + 0.5$$

To see a complete development of this reasoning see Penadés (2000):83–120.

⁸ A full account of the mathematical development is offered in Penadés (2000):179–232.

⁵ This restriction is important. If $V_{S_d+1}^{nec} \leq V_{S_d}^{suf}$, then the necessary condition to take $S_d + 1$ seats would lie within the range $[V_{S_d+1}^{nec}, V_{S_d}^{suf}]$ and conclusion 3 would no longer hold.

3.1. Quota-based methods

$$V_{S_d}^{nec}(M_d, P_d, n) = \frac{P_d(S_d - 1) + 1 + n}{P_d(M_d + n)} \quad (1)$$

where $1 \leq S_d \leq M_d$ and $n > -M_d$

$$V_{S_d}^{suf}(M_d, P_d, n) = \begin{cases} \frac{P_d(S_d - 1) + P_d - 1 + n}{P_d(M_d + n)} & \text{if } S_d \leq M_d - P_d + 2 \\ \frac{S_d - 1}{M_d + n} + \frac{M_d - S_d + 1 + n}{(M_d - S_d + 2)(M_d + n)} & \text{if } S_d \geq M_d - P_d + 2 \end{cases} \quad (2)$$

where $0 \leq S_d \leq M_d - 1$

Threshold functions enable us to establish a general formula for the inclusion and exclusion threshold:

$$V_{S_d=1}^{nec}(M_d, P_d, n) = \frac{n + 1}{P_d(M_d + n)} \quad (3)$$

If $V_d^p < V_{S_d=1}^{nec}(M_d, P_d, n)$, then it is impossible for party p to obtain a seat in the parliament.

$$V_{S_d=1}^{suf}(M_d, P_d, n) = \begin{cases} \frac{P_d - 1 + n}{P_d(M_d + n)} & \text{if } P_d \leq M_d + 1 \\ \frac{1}{(M_d + 1)} & \text{if } P_d \geq M_d + 1 \end{cases} \quad (4)$$

If $V_d^p > V_{S_d=1}^{suf}(M_d, P_d, n)$, then a party p will necessarily obtain at least one seat in the parliament.

3.2. Divisors-based methods

$$V_{S_d}^{nec}(M_d, P_d, c) = \frac{S_d - 1 + c}{M_d - 1 + P_d c} \quad (5)$$

if $c \geq 0$ and $1 \leq S_d \leq M_d$ ⁹

⁹ It is interesting to note the functioning of this expression when $c = 1$, that is to say, when the d'Hondt electoral formula is used. In this case, the necessary value to win k seats equals the necessary value to win 1 seat multiplied precisely by k times. This does not happen when other formulae like Sainte-Lagué ($c = 0.5$) or even the quota-based electoral formulae are used.

$$V_{S_d}^{suf}(M, P_d, c) = \begin{cases} \frac{S_d - 1 + c}{M_d + 1 + P_d(c - 1)} & \text{if } S_d \leq M_d - P_d + 2 \text{ and } 0 \leq c \leq 1 \\ \frac{S_d - 1 + c}{(M_d + 1)c + S_d(1 - c) - 1 + c} & \text{if } S_d \geq M_d - P_d + 2 \text{ and } 0 \leq c \leq 1 \\ \frac{S_d - 1 + c}{M_d - 1 + 2c} & \text{if } c \geq 1 \end{cases} \quad (6)$$

where $0 \leq S_d \leq M_d - 1$.

Finally¹⁰, the general form of both the threshold of representation and the threshold of exclusion for divisor-based methods are:

$$V_{S_d=1}^{nec}(M_d, P_d, c) = \frac{c}{M_d - 1 + P_d c} \text{ if } c \geq 0 \quad (7)$$

and

$$V_{S_d=1}^{suf}(M_d, P_d, c) = \begin{cases} \frac{c}{M_d + 1 + P_d(c - 1)} & \text{if } P_d \leq M_d + 1 \text{ and } 0 \leq c \leq 1 \\ \frac{1}{M_d + 1} & \text{if } P_d \geq M_d + 1 \text{ and } 0 \leq c \leq 1 \end{cases} \quad (8)$$

Penadés' contribution to this field derives not only from his success in providing generally applicable inclusion and exclusion thresholds, but also in presenting a function that is capable of calculating the necessary and sufficient number of votes required to win any number of seats. Another important point must be noted in this respect: these functions are conceived for cases in which there is uncertainty about the distribution of votes between all political parties. If we know how the votes are distributed between all competing parties,

¹⁰ At this point, it is necessary to make a number of observations about the importance of the legal threshold, T_L . Some electoral laws establish a minimum percentage of votes required to win parliamentary representation. These types of electoral barrier, which are largely intended to prevent the entry of minority parties into parliament, are known as legal thresholds. This variable is, of course, of great importance when considering the necessary and sufficient conditions to obtain S_d seats. If $T_L > V_{S_d}^{suf} > V_{S_d}^{nec}$ then, T_L gives us the share of votes that are required to win S_d seats.

that is, after the elections have taken place and once the results have been announced, it is possible to make a straightforward calculation of the number of seats that each party will get. Threshold functions indicate the range of votes within which a determined number of seats can be won. The exact distribution of seats can only be calculated once we know the distribution of votes among the competing parties.

As I pointed out before, these functions are calculated at district level. A logical question must follow, then: is it possible to find a set of functions capable of calculating the necessary and sufficient number of votes to win a given number of seats at the national level? The question extends the analysis to an aggregated level. How much nationwide support does a party need to win its first seat in the parliament? And the majority of the seats? In general, what are the necessary and sufficient shares of the vote nationwide that a party needs to win any number of seats, considering all districts in which the country is divided?

4. Aggregated threshold functions

There have been some attempts to find out a function capable of producing nationwide thresholds for winning one seat. Major contributions on this issue are mainly due to Taagepera (1998, 2002)¹¹. However, his approach has some of the problems I discussed above, particularly the emphasis on calculating thresholds for just one seat and considering just a few electoral formulae. In this section, I go beyond Taagepera's and Penadés' approaches. The goal is to calculate aggregated threshold functions that can provide nationwide threshold values for any number of seats in any electoral system.

Briefly, seats are allocated according to the share of votes that each party wins. In other words, the total number of seats won nationwide by a party, S_T^p , is a function of its total share of votes, V_T^p . Formally,

$$f : V_T^p \rightarrow S_T^p \quad (9)$$

Aggregation means the addition of individual values. Since threshold functions are defined at district level for any elemental electoral system, aggregated threshold functions will require a new definition of the electoral system. I define a complete electoral

system as a set of rules with the following elements: an electoral formula, c or n , the number of districts in which the country is divided, D , a $1 \times D$ vector with all district magnitudes, $\mathbf{M}_d$ and the number of seats in the legislative assembly, M . This definition must be complemented, though, with the number of parties competing in all districts, P . Before I introduce the set of aggregated threshold functions, some notation to define concepts and introduce the methodology of aggregation is required.

Notation 1. The aggregated threshold functions will be based on both $V_{S_d}^{nec}(F, M_d, P_d)$ and $V_{S_d}^{suf}(F, M_d, P_d)$ as defined in the previous section. The reason for proceeding in this way is that the purpose of these functions is to make it possible to calculate the intervals of votes that will define the necessary and sufficient conditions for any party to win S_T^p seats at the aggregate level.

Notation 2. All parties are distributed in a $1 \times P$ vector so that $\mathbf{P} = [1, 2, \dots, P]$ where P is the total number of parties competing in the whole territory. Since a distinctive feature of any democracy is electoral competition, then $P \geq 2$.

Notation 3. The number of districts, D , is a parameter that must also be taken into account. The number of districts refers to all the constituencies in which the territory is divided. This number ranges from 1 — the whole country is a single constituency (e.g. Moldova) to M , the size of the assembly — single-member constituencies (e.g. United Kingdom).

Notation 4. Since electoral territories are divided into districts, a vector of size $1 \times D$, $\mathbf{M}_d$, is defined as the vector that contains all district magnitudes in the territory, $\mathbf{M}_d = [M_1, \dots, M_D]$, where M_d refers to the size of district d . Also note that

$$\sum_{d=1}^D M_d = M \quad (10)$$

where M refers to the total number of seats in the parliament.

Notation 5. Whatever the number of districts in which the country is divided or even the numbers of tiers, seats are won at district level¹². Hence, we can define another

¹¹ The questions that Taagepera posits are similar to the one proposed here: is there a function that calculates nationwide electoral thresholds for winning one seat? Taagepera pursues an aggregation method using a reasoning he previously used at district level. His attempt introduces the importance of variables such as the size of the assembly or the number of districts.

¹² In mixed electoral systems, think for example in a country that chooses half of its parliament in single-member districts and half of the parliament in a single district, both tiers are allocating seats at district level, one district being uninominal, the other one having the size $M/2$.

$1 \times D$ vector, $\mathbf{S}_j$, this indicating a particular distribution of seats won in each district. Formally, this vector is expressed as $\mathbf{S}_j = [S_{1j}, \dots, S_{Dj}]$, where S_{dj} indicates the number of seats won in district d in a particular distribution of seats, j . Since aggregation is understood as the sum of individual values, the aggregated value of seats, S_T , is the sum of all the elements that form the vector $\mathbf{S}_j$. So,

$$S_T = \sum_{d=1}^D S_{dj} \quad (11)$$

provided that

$$0 < S_{dj} \leq M_d \quad (12)$$

and

$$0 < S_T \leq M \quad (13)$$

Notation 6. It should also be noted that two or more particular combinations of seats, $j, k, \dots, z$, may have the following property:

$$\sum_{d=1}^D S_{dj} = \sum_{d=1}^D S_{dk} = \dots = \sum_{d=1}^D S_{dz} = S_T \quad (14)$$

In other words, different combinations of seats won in the same distribution of districts, $\mathbf{M}_d$, may produce the same aggregated number of seats, S_T .

Notation 7. The total share of votes won by party p , V_T^p , can also be disaggregated in a $1 \times D$ vector compounded by all individual shares of the vote won in each district. Thus, $\mathbf{V}^p = [V_1^p, \dots, V_D^p]$, where V_d^p refers to the share of votes won by party p in district d and subject to

$$V_T^p = \sum_{d=1}^D V_d^p \quad (15)$$

Notation 8. In the light of *Notations 4, 5 and 7*, electoral formulae are functions that allocate seats according to the share of the vote won by each party, thus, $\forall \mathbf{V}^p \exists! \mathbf{S}_j^p$. So, for every particular distribution of votes among all districts that each party obtains, $\mathbf{V}^p$, there is one and only one particular distribution of seats, $\mathbf{S}_j^p$ which produces an aggregated number of seats, S_T^p .

Notation 9. Given that districts may have different magnitudes, a measure to weigh each district must be

incorporated in the function. This measure is based upon the size of the parliament, M , and has the following form:

$$\text{Weight} = \frac{M_d}{M} \quad (16)$$

Condition 1. In this article, it is assumed that all district magnitudes are commensurate with their voting population. In other words, magnitudes are designed in line with a ratio between voters and seats. Hence, no malapportionment effect is taken into consideration.

Condition 2. Finally, and for simplicity, it is assumed that the number of competing parties is the same in all districts. Formally,

$$\mathbf{P}_1 = \dots = \mathbf{P}_D \text{ for any } d \in \mathbf{M}_d \quad (17)$$

where $\mathbf{P}_d$ refers to the vector of parties competing in district d .

Once terms have been defined and the method of aggregation introduced, aggregated threshold functions are defined as follow.

Definition 3. $V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$: Given a particular distribution of seats, $\mathbf{S}_j$, won in a distribution of districts $\mathbf{M}_d$, the size of the parliament, M , the electoral formula, F , and the number of competing parties, P , $V_{S_T}^{nec}$ defines the minimum number of votes that a party needs to get S_T seats distributed according to $\mathbf{S}_j$. So, if a party expects to win those S_T seats distributed according to $\mathbf{S}_j$, its total share of the vote must be at least equal to $V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$.

Definition 4. $V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$: Given a particular distribution of seats $\mathbf{S}_j$ won in a distribution of districts $\mathbf{M}_d$, $V_{S_T}^{suf}$ defines the sufficient condition for a party to obtain S_T seats distributed according to $\mathbf{S}_j$ in an electoral system with a parliament of size M , a number of districts D , an electoral formula F and a number of competing parties, P . To be sure of winning those S_T seats distributed according to $\mathbf{S}_j$, a party must win a share of votes higher than $V_{S_T}^{suf}(M, D, \mathbf{S}_j, F, P)$.

From *definitions 3, and 4* the following conclusions can be inferred.

Conclusion 4. If $V_T^p < V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$, then $S_T^p < S_T$.

Note from *Notation 6*, however, that two or more particular distributions of seats, $\mathbf{S}_j, \mathbf{S}_k, \dots, \mathbf{S}_z$, may

produce the same total number of seats, S_T . So if $V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_k, F, P) < V_T^p$, then party p will fulfil the minimum condition to win S_T seats but distributed according to $\mathbf{S}_k$.

Conclusion 5. If $V_T^p > V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$, then $S_T^p \geq S_T$

Conclusion 6. If $V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P) \leq V_T^p \leq V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$, then $\max(S_T^p) = S_T$

Finally, aggregated threshold functions can be expressed mathematically as:

$$V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P) = \sum_{d=1}^D \frac{M_d}{M} V_{S_d}^{nec} \quad (18)$$

and

$$V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P) = \sum_{d=1}^D \frac{M_d}{M} \min(V_{S_d}^{suf}, V_{S_{d+1}}^{nec}) \quad (19)$$

Looking at expressions (18) and (19), one can see that aggregated threshold functions are made of two components. First, the aggregation term made of the sum of all individual values and, second the weighted value of the necessary or sufficient shares of the vote for a given number of seats. Once aggregated threshold functions are introduced, another remark must be made. Again, the values provided by these functions do not offer the exact number of seats that each party wins. As I pointed out in the previous section, aggregated threshold functions offer the best and the worst conditions to obtain a determined number of seats but they do not allocate seats to each party. The exact distribution of seats depends upon how the whole vote is shared by all parties and that is something that can only be known once an election has occurred.

As defined here, aggregated threshold functions have the following properties. First, they are universal since they can be applied to any conceivable electoral system. This property was also enjoyed by threshold functions announced by Penadés (2000) as I showed in the previous section. A second and exclusive property of aggregated threshold functions is, however, its generality. Aggregated threshold values are general in the sense that they take into account not the functioning of the electoral system at a given district, but taking all districts into account.

These two properties and specially the second one, generality, allow summarizing the mechanical performance of any electoral system in a single value. Either the necessary or the sufficient shares of votes are measures of how electoral systems behave as a whole and therefore they can be used to study the consequences

of any electoral system. Aggregated threshold functions produce a single value which is exclusive for every electoral system that completely and parametrically summarizes the mechanical performance of that electoral system¹³.

5. Data validation

The main purpose of this section is to test the validity of the aggregated threshold functions. The idea here is to check whether the total share of the vote that produces the total number of seats won by political parties in a number of countries accords with the conclusions reached in the previous section. In order to carry out this test, I have analyzed the electoral results obtained by the main political parties in general elections in several countries. The selection of these elections does respond to a double logic. First, the selected cases show different institutional designs so that a broader test about the validity of aggregated threshold functions can be obtained. Second, the application of aggregated threshold functions requires district data, so the cases are also selected depending on the availability of this type of data. The sample of cases could be broader (Ruiz-Rufino, 2005a,b); however, as I mentioned before, the purpose of this article is to introduce in an analytical way a model to measure the performance of an electoral system and the main goal of this section is just to check whether the total share of the vote that produces the total number of votes won by political parties in some electoral processes fulfils the conclusions derived in the previous section. For this reason, I have selected these cases randomly.

To cope with the maximum number of possible options, I have focused on some elections that have occurred under different electoral systems. In the case of divisor-based electoral systems, I use the results obtained by the main political parties in the 1979 general election in Spain, the 1991 and 1994 general election

¹³ One interesting application of this can be seen in the following case: given a complete electoral system, the combination of seats that produce the lowest necessary aggregated value for half of the seats in the parliament can be calculated for any party, p . This value is understood as the share of the national vote that any party must win in order to obtain that number of seats; below this value, it is impossible to win that majority of seats in the parliament. If this value is contrasted with the point of perfect proportionality, one can see how much that complete electoral system deviates from such an ideal point. By applying aggregated threshold functions in this manner, it is obtained, then, a straightforward way to test the mechanical performance of any complete electoral system using a single value.

in Bulgaria and the 1994 general elections in Moldova. To test the validity of the aggregated threshold functions in quota-based electoral systems, I use the results won by the main political parties that participated in the 1986 general elections in Costa Rica, the 1997 general elections in Honduras and the 1999 general elections in Benin. Finally, to test the validity of the majoritarian electoral systems, I check the 1993 general election in Canada and the 1997 general election in the United Kingdom. In total, these amount to 53 observations with sufficient variation to be in a fair position to validate the results produced by aggregated threshold functions.

The method used to validate the aggregated threshold values is that described in the previous section. For each party there is a vector containing all votes won in each district and another vector with all the seats also obtained in each district. If we add up all the shares of the vote won in each district, we obtain the total share of the vote for that party. The same operation is performed for the vector containing the number of seats won in each district. On the basis of this information and the remaining variables for each complete electoral system, I have applied $V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$ and $V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$. To test the functions and for practical reasons, I will use the effective number of parties as a proxy of the total number of competing parties¹⁴.

The data validation for divisor-based electoral systems is shown in Table 1. As noted above, the countries to which aggregated threshold functions have been applied are Spain, Bulgaria and Moldova. These three countries are different from each other. Whereas Spain

Table 1

Results for general elections in three countries with divisors-based electoral systems

Country	Election	Party	$V_T^p(\%)$	S_T^p	$V_{S_T}^{nec}$	$V_{S_T}^{suf}$
Spain	1979	UCD	34.84	168	31.96	41.27
Spain	1979	PSOE	30.4	121	24.71	30.62
Spain	1979	PCE	10.77	23	5.33	6.11
Spain	1979	CD	6.05	10	2.04	2.32
Spain	1979	CiU	2.69	8	1.89	2.14
Spain	1979	PNV	1.65	7	1.41	1.77
Spain	1979	PSA-PSA	1.81	5	1.06	1.29
Spain	1979	HB	0.96	3	0.63	0.77
Spain	1979	UN	2.11	1	0.26	0.28
Spain	1979	ERFN	0.69	1	0.26	0.28
Spain	1979	EE	0.48	1	0.20	0.25
Spain	1979	C-UPC	0.33	1	0.19	0.25
Spain	1979	PAR	0.21	1	0.20	0.25
Bulgaria	1991	SDS	34.36	110	33.37	40.92
Bulgaria	1991	BSP	33.14	106	31.36	39.04
Bulgaria	1991	DPS	7.55	24	6.94	8.76
Bulgaria	1994	BSPASEK	43.50	125	37.88	45.90
Bulgaria	1994	SDS	24.23	69	21.75	25.77
Bulgaria	1994	BZNS-DP	6.51	18	5.74	6.77
Bulgaria	1994	DPS	5.44	15	4.30	5.37
Bulgaria	1994	BBB	4.73	13	4.21	4.91
Moldova	1994	PDAM	43.18	56	41.18	53.33
Moldova	1994	PSMUE	22	28	20.59	26.67
Moldova	1994	BTI	9.21	11	8.09	10.48
Moldova	1994	AFPCD	7.53	9	6.62	8.57

Data Source — Spain — <http://www.elecciones.mir.es/>.

Data Source — Moldova and Bulgaria — <http://www.essex.ac.uk/elections>.

and Bulgaria have a large number of districts, 52 and 31, respectively, Moldova has a single district.

Formally,

$$V_T^p \in [V_{S_T}^{nec}, V_{S_T}^{suf}] \forall S_T^p \quad (20)$$

This idea must, however, be properly understood. What I intend to confirm here are conclusions 4, 5 and 6 referred above. Recall that conclusion 4 established that

$$\text{If } V_T^p < V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P), \text{ then } S_T^p < S_T \quad (21)$$

Consider the following illustration. The PNV party won seven seats in the 1979 general election in Spain. The necessary proportion of votes required to win those seven seats distributed exactly in the way that PNV won them is 1.41% of the vote at national level. Since PNV won 1.65% that minimum requirement was fulfilled.

Conclusion 5 showed something rather different. Formally,

$$\text{If } V_T^p > V_{S_T}^{suf}(M, \mathbf{M}_d, \mathbf{S}_j, F, P), \text{ then } S_T^p \geq S_T \quad (22)$$

What this expression indicates is the possibility for a party to win a higher number of seats when the total

¹⁴ Following Laakso and Taagepera (1979), the effective number of parties (ENP) can be calculated using the following formula:

$$ENP = \sum \frac{1}{(V_T^p)^2}$$

where V_T^p is the share of total votes won by party p . The use of the effective number of parties as a proxy of the total number of competing parties may be controversial. The ENP assumes a concrete spatial distribution of the vote whereas in terms of aggregated threshold functions, the number of competing parties refers to those political parties that have a chance to win at least one seat. I am aware of this problem here and therefore in this study, the ENP is interpreted not as the result of a particular spatial distribution of votes but as proxy that measures the number of parties with real options to win political representation. Other measures such as the total number of competing parties would produce misleading results given that the number of those competing in a district can be very high. Again, aggregated threshold functions produce more sensible results when the number of parties with real options to win a seat is incorporated into the function.

share of the vote that this party obtains is higher than the sufficient proportion of votes to win S_T seats distributed according to a particular combination $\mathbf{S}_j$. In other words, a higher share of the vote won by party p may produce a different combination of seats that may also produce a higher number of the original total seats for which $V_{S_T}^{suff}(M, \mathbf{M}_d, \mathbf{S}_j, F, P)$ was first applied. Look at the data for the PCE party in Table 1. This party won 10.77% of the national vote and obtained 23 seats in the 1979 general election in Spain. The proportion of sufficient votes to win those 23 seats distributed in all districts exactly in the way that PCE won them is 6.11% that is to say; PCE won a share of the total vote 4.66 percentage points higher than the sufficient proportion of votes to win those 23 seats. However, PCE could only win 23 seats at most because given the distribution of those seats in the districts where they were won, the sufficient proportion of votes was 6.11%. PCE wasted, then, those 4.66 percentage points of the votes. This party could have won a different number of total seats if a different combination of seats were produced¹⁵.

Finally, conclusion 6 established that,

$$\text{If } V_{S_T}^{nec}(M, \mathbf{M}_d, \mathbf{S}_j, F, P) \leq V_T^p \leq V_{S_T}^{suff}(M, \mathbf{M}_d, \mathbf{S}_j, F, P), \\ \text{then } \max(S_T^p) = S_T \quad (23)$$

This reasoning follows from the discussion just mentioned about the sufficient proportion of votes. Consider in this case the electoral results obtained by the Bulgarian political party BZNS-DP in the 1994 election. This party won 6.51% of the national vote and obtained 18 seats. When aggregated threshold functions are applied to the combination of seats that produced those 18 seats for this party it is observed that 5.74% is the necessary proportion of votes and 6.77% the sufficient proportion. The share of votes won by BZNS-DP is included in this interval and therefore the maximum number of seats that can be won are precisely those 18 seats distributed according to $\mathbf{S}_j$.

This logic can also be observed when other electoral systems are analyzed. Table 2 shows data for three different quota-based electoral systems. All the countries included in this table used the Hare quota electoral formula in their electoral system, but they differed in terms of the number of districts and size of the assembly. So, whereas Costa Rica had seven districts and an assembly

Table 2

Results for general elections in three countries with quota-based electoral systems

Country	Election	Party	$V_T^p(\%)$	S_T^p	$V_{S_T}^{nec}$	$V_{S_T}^{suff}$
Costa Rica	1986	LN	47.83	29	43.55	45.93
Costa Rica	1986	US	41.44	24	34.78	37.15
Costa Rica	1986	PU	2.70	1	0.71	1.05
Costa Rica	1986	AL	0.32	1	0.71	1.05
Costa Rica	1986	UA	1.15	1	0.71	1.05
Costa Rica	1986	AP	2.43	1	0.71	1.05
Honduras	1997	PL	49.54	65	43.43	45.64
Honduras	1997	PN	41.56	56	35.47	37.82
Honduras	1997	PINU-SD	4.13	4	1.29	1.84
Honduras	1997	DC	2.61	2	0.64	0.92
Honduras	1997	UD	2.14	1	0.32	0.46
Benin	1999	RB	22.69	27	21.45	28.27
Benin	1999	PRD	12.17	11	6.61	11.57
Benin	1999	FARD	5.49	10	4.29	9.76
Benin	1999	PSD	9.27	9	6.41	8.84
Benin	1999	MADEP	9.21	6	2.80	5.62

Data source — Costa Rica — Tribunal Supremo de elecciones <http://www.tse.go.cr/>.

Data Source — Honduras — Tribunal Supremo Electoral <http://www.tne.hn/>.

Data Source — Benin — Dissou (2002).

with 57 members in the 1986 general election, Benin had 24 districts and an assembly of 84 members for the 1999 election and Honduras 18 districts where a total of 128 seats in the assembly were elected in the 1997 general election. As in the case of divisor-based electoral systems, the aggregated threshold values give us an interval in which all shares of the vote can be located. So, the R.B. party won 27 out of 84 seats of the assembly with 22.69% of the total vote in the 1999 general election in Benin and the threshold functions predicted that, in this case, this number of seats so particularly distributed could not be won by a party obtaining less than 21.45% of the vote.

Finally, Table 3 shows the electoral results for two majoritarian electoral systems. Information is given for the 1997 general election in the United Kingdom (U.K.) and for the 1993 general election in Canada. The predictions of the aggregated threshold functions seem to be accurate at least in terms of the necessary number of votes to win S_T seats. No parties won S_T^p seats with a share of the vote below that predicted by the aggregated threshold functions.

Aggregated threshold functions seem, therefore, to constitute a convincing measure through which one can measure the mechanical performance of an electoral system.

6. Conclusions

In this paper, I have defined a measure to test the performance of any electoral system. Aggregated threshold

¹⁵ In other words, given the size of the districts where PCE won representation, the spatial distribution of voters produced at most the result obtained in the 1979 Spanish election. If the PCE's voters had been distributed differently, maybe, the proportion of wasted votes would have been lower and a different (maybe higher) number of seats might have been won.

Table 3
Results for general elections in two majoritarian electoral systems

Country	Election	Party	V_T^p (%)	S_T^p	V_{ST}^{nec}	V_{ST}^{suf}
U.K.	1997	Labour	43.2	418	19.58	31.71
U.K.	1997	Conservatives	30.6	165	7.72	12.51
U.K.	1997	Lib-Dems	16.7	46	2.15	3.49
U.K.	1997	SNP	2	6	0.28	0.45
U.K.	1997	UU	0.83	10	0.46	0.75
U.K.	1997	SDLP	0.61	3	0.14	0.22
U.K.	1997	PC	0.51	4	0.18	0.30
Canada	1993	LP	41.3	177	15.33	30
Canada	1993	BQ	13.5	54	4.67	9.15
Canada	1993	RP	18.7	52	4.50	8.81
Canada	1993	NDP	6.9	9	0.77	0.33
Canada	1993	PCP	16.0	2	0.17326	0.33

Data source — U.K. — www.electoralcommission.org.uk.

Data source — Canada — www.elections.ca.

functions show the necessary and sufficient share of votes nationwide to win any number of seats and for any complete electoral system. These functions have two interesting properties. First, they are universal since they can be applied to any conceivable institutional design. Second, aggregated threshold functions are also general in the sense that they can be calculated taking into account all districts in which a country is divided. As this paper shows, a measure with these two properties goes beyond the commonly used values to test the performance of electoral systems that exist in the literature.

Aggregated threshold functions are informative. They do not offer the share of votes to win a given number of seats but a range of votes among which a number of seats distributed according to a particular combination can be obtained. As the data shows, these ranges of votes change depending on the institutional design and the size of the party. Considering large parties, the range of aggregated threshold values is wider in single-member district than in divisor-based or quota-based electoral systems. It is also interesting to observe, that some parties are winning seats with votes close to the necessary values while others with votes close to the sufficient values. This may suggest that some parties may be more aware of the mechanical functioning of the electoral system where they compete than others and they have their strength more efficiently distributed. These are open empirical questions left for future research.

As it has been said here, a major implication of using aggregated threshold functions is that they can be used to produce a single exclusive value that parametrically summarizes the mechanical functioning of any electoral system. Electoral reformers or any student of electoral systems may find in this value a useful tool. Aggregated threshold functions may help electoral reformers to see

what mechanical performance they can expect in every electoral system. Student of electoral systems may also have in these functions a way to characterize any electoral system. Aggregated threshold functions offer continuous values so that every complete electoral system can be located in a continuum. Once in a continuum, electoral systems can be contrasted with an ideal point like perfect proportionality having, therefore, a straight way to compare electoral institutions.

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